Entanglement Entropy of Systems with Spontaneously Broken Continuous Symmetry

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Abstract

We study entanglement properties of systems with spontaneously broken continuous symmetry. We find that in addition to the expected area law behavior, the entanglement entropy contains a subleading contribution which diverges logarithmically with the system size in agreement with the Monte Carlo simulations of A. Kallin *et. al.* (arXiv:1107.2840). The coefficient of the logarithm is a universal number given simply by $\frac{N_G (d-1)}{2}$ where N_G is the number of Goldstone modes and d is the spatial dimension. This term is present even when the subsystem boundary is straight and contains no corners, and its origin lies in the interplay of Goldstone modes and restoration of symmetry in a finite volume. We also compute the "low-energy" part of the entanglement spectrum and show that it has the same characteristic "tower of states" form as the physical low-energy spectrum obtained when a system with spontaneously broken continuous symmetry is placed in a finite volume.

I.   INTRODUCTION

In recent years there has been a lot of theoretical interest in entanglement properties of quantum states of matter. Entanglement has proved to be a useful probe of non-local correlations for both gapless and gapped systems. The most commonly used characterization of entanglement is the entanglement entropy $S = -tr(\rho_A \log \rho_A)$, defined as the von Neumann entropy associated with the reduced density matrix ρ_A of a subsystem. A close relative of the entanglement entropy is the Renyi entanglement entropy $S_n = -\frac{1}{n-1} \log tr(\rho_A^n)$.

To date, the most impressive progress has been made in the study of entanglement entropy of one dimensional critical states. Here, it has been shown[1,2] that for systems described by conformal field theories (CFT's) the entanglement entropy of a system of length L behaves as,

$$S = \frac{c}{3} \log L/a + \gamma \tag{1.1}$$

where a is the short distance cut-off. The coefficient c is a universal number known as the central charge of the CFT. The subleading constant γ depends on the system geometry (*e.g.* the ratio of the subsystem size to the full system size). Although γ is not fully universal as is clear from the cut-off dependence of (1.1), it is universal up to an additive constant. The universal behavior of the entanglement entropy (1.1) has proved useful for extraction of the central charge of the governing CFT in numerical density-matrix renormalization group (DMRG) studies of one-dimensional critical systems.

Our present understanding of entanglement in dimension $d > 1$ is far less complete. However, it is generally expected that for both critical and non-critical systems the leading contribution to the entanglement entropy scales as the area of the subsystem boundary, $S = C\mathcal{A}/a^{d-1}$.[1] This "area law" contribution is related to short-range entanglement in the vicinity of the boundary, and as a result, the proportionality constant C is non-universal. However, for critical scale invariant systems one expects a subleading, fully universal, geometric contribution γ to the entanglement entropy,

$$S = C\mathcal{A}/a^{d-1} + \gamma \tag{1.2}$$

The scaling form (1.2) is believed to hold for scale invariant systems in $d = 2$ with arbitrary smooth boundaries and for systems in $d = 3$ whose boundary is flat.[5–8] Additional logarithmic contributions are expected in $d = 2$ if the boundary has sharp corners and in $d = 3$ if the boundary is curved. We note that all the above stated results/hypotheses on the entanglement entropy apply also to the Renyi entanglement entropy S_n, with the constants C and γ acquiring a dependence on n.

The scaling hypothesis (1.2) relies on the following argument. The entanglement entropy, being a dimensionless quantity, can only depend on ratios of length (or energy) scales.

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[1] A notable exception is provided by systems with a Fermi-surface, where the area law receives a multiplicative logarithmic correction.[3,4]

However, in a scale invariant theory, the only two length scales are the total system size L and the short distance cut-off a, with the corresponding energy scales $1/L^z$ and $1/a^z$, where z is the dynamical critical exponent. Thus, any dependence of the entanglement entropy on the system size must come together with the dependence on the short distance cut-off. The variation $a\frac{dS}{da}$ comes from short-range entanglement in the vicinity of the boundary and is expected to take the form of some local geometric quantity integrated over the boundary area. If the boundary is straight and has no corners, there are no non-trivial local geometric quantities, and the only possible cut-off dependent contribution to the entanglement entropy is proportional to the integral of 1 over the boundary, *i.e.* the boundary area, in accord with Eq. (1.2). On the other hand, for a two-dimensional boundary with corners each corner can contribute a constant to $a\frac{dS}{da}$ resulting in an entanglement entropy which depends logarithmically on a and hence on L. For a generalization of these arguments to curved boundaries, see Refs. 9–11.

In a recent breakthrough it has become possible to extract the Renyi entanglement entropy of quantum systems using Monte-Carlo simulations.[12] One of the first applications of the method of Ref. 12 has been to study the entanglement entropy of a spin-1/2 Heisenberg model on a two-dimensional square lattice.[13] The Monte-Carlo simulations were performed using a torus geometry, with the subsystem being either a cylinder or a square. As expected, the leading contribution to the Renyi entanglement entropy was found to scale linearly with the system size. However, surprisingly, a subleading logarithmic correction was observed for both geometries studied.

The ground state of the Heisenberg model on a two-dimensional square lattice is known to spontaneously break the $SU(2)$ spin-rotation symmetry to a $U(1)$ subgroup. Thus, in the infinite volume limit the ground state is infinitely degenerate and the ground state manifold is a two-dimensional sphere, whose points are labeled by the orientation of the Néel order parameter $\vec{n}$. The low energy excitations above a particular ground state are two linearly dispersing Goldstone bosons, known as spin-waves. The interactions between the spin-waves are irrelevant in the low-energy limit and one, thus, simply obtains a theory of two free scalar bosons. Naively, this theory is scale invariant and so according to Eq. (1.2), one does not expect any subleading logarithmic contributions to the entanglement entropy as long as the boundary of the subregion is smooth, *e.g.* for the cylindrical subregion geometry. For the square subregion geometry, one could attribute the logarithmic correction to the corners of the boundary, however, the Monte-Carlo estimate of the coefficient of the logarithm differs both in sign and by one order of magnitude compared to the previous calculation based on a free bosonic field theory.[8]

The caveat to the above discussion is that a system with a spontaneously broken continuous symmetry is, in fact, not scale invariant at low energy in the same way that a CFT is. The reason for this is that symmetry is always restored in a finite volume and instead of the degenerate vacuum manifold one has a unique ground state and a "tower" of excited states carrying a definite charge under the symmetry group. For instance, in the case of the Heisenberg model with its $SU(2) \to U(1)$ symmetry breaking, the tower of states can be

described by an effective Hamiltonian,

$$H_{\text{tower}} = \frac{c^2 \vec{S}^2}{2\rho_s V} \tag{1.3}$$

where $\vec{S}$ is the total spin of the system, c is the spin-wave velocity, ρ_s is the spin-stiffness and V is the volume of the system. Note that the spacing between the energy levels of the tower scales as $V^{-1} = L^{-d}$. For system dimension $d > 1$, where spontaneous breaking of continuous symmetry is possible, this spacing is much smaller than the spin-wave gap c/L.

Thus, in a system with spontaneous breaking of continuous symmetry, to a given length scale L there correspond two infra-red energy scales: the tower of states spacing $c^2/(\rho_s L^d)$ and the spin-wave gap c/L. In general, we expect the universal part of the entanglement entropy ΔS to depend on the ratio of these energy scales:

$$S = C \frac{\mathcal{A}}{a^{d-1}} + \Delta S(\rho_s L^{d-1}/c) \tag{1.4}$$

Here, the function ΔS a priori can depend on the system geometry. Note that the argument of ΔS is much greater than one.[2] In this paper, we argue that for a system with $O(N) \rightarrow O(N-1)$ symmetry breaking, ΔS behaves as

$$\Delta S = b \log(\rho_s L^{d-1}/c) + \gamma_{\text{ord}}, \quad b = \frac{N-1}{2} \tag{1.5}$$

As usual, Eq. (1.5) assumes smooth boundaries in $d = 2$ and flat boundaries in $d = 3$. Thus, the coefficient of the logarithmic divergence with system size L is directly expressed in terms of the number of Goldstone modes $N - 1$. In particular, for the case of the Heisenberg antiferromagnet $N = 3$, $b = 1$. We note that in addition to the logarithmic correction in Eq. (1.5) there also appears a finite constant γ_{ord} that is generally expected to depend on the system geometry. Unlike in the case of entanglement entropy in one dimension, Eq. (1.1), where the presence of a logarithm rendered γ universal only up to an additive contribution, here γ_{ord} is fully universal, as all the short-distance physics is absorbed into the order-parameter stiffness ρ_s and the Goldstone velocity c. We will present an expression for γ_{ord} in terms of a related constant in a free scalar theory. We numerically evaluate the latter constant (hence obtaining γ_{ord}) using the correlation matrix technique of Ref. 14 for the geometry studied in the Monte-Carlo simulations of Ref. 13.

In addition to the entanglement entropy, we compute the Renyi entropies S_n, which are

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[2] Clearly, for $L \rightarrow \infty$, the quantity $\rho_s L^{d-1}/c \gg 1$. One can, however, ask if this quantity can become of $O(1)$ or smaller in a system close to a continuous phase transition into an antiferromagnetically disordered phase where $\rho_s \rightarrow 0$. For the effective low-energy description of the antiferromagnet to apply, we need the system size L to be much larger than the correlation length of the system ξ, thus, $\rho_s L^{d-1}/c \gg \rho_s \xi^{d-1}/c$. A priori, there need not be a relation between ρ_s and ξ. However, for all quantum phase transitions known to the authors, $\rho_s \xi^{d-1}/c$ either goes to a constant at the transition or diverges.

also found to satisfy the scaling form in Eqs. (1.4),(1.5). The coefficient of the area law C is now expected to depend on the replica index n and so is the finite size constant γ. However, the coefficient of the logarithmic correction b is found to be independent of the replica index.

We also study the full low energy spectrum of the entanglement Hamiltonian H_E defined in terms of the reduced density matrix of the subsystem ρ_A as $\rho_A = \exp(-H_E)$. We show that the low energy part of H_E up to a constant energy shift satisfies $H_E = H_{\text{tower}}/T_E$, with H_{tower}, Eq. (1.3), the physical tower of states Hamiltonian of the subsystem. The "effective temperature" T_E is found to scale with the system size as $T_E \sim c/L$. These findings are remarkably similar to recent results on gapped topological phases, which possess gapless edge states.[15–22] It was found that for such phases the entanglement Hamiltonian associated with a subsystem is proportional to the actual Hamiltonian governing the edge states that would be exposed if a physical cut was performed along the subsystem boundary.

Our conclusions are based on two calculations. The first involves a simple quantum mechanical model of two coupled quantum $O(N)$ rotors $\vec{n}_A$ and $\vec{n}_B$, representing the average order parameter in each subsystem,

$$H = \frac{c^2 \vec{L}_A^2}{2\rho_s V_A} + \frac{c^2 \vec{L}_B^2}{2\rho_s V_B} - J\vec{n}_A \cdot \vec{n}_B \qquad (1.6)$$

Here $\vec{L}_{A,B}$ is the angular momentum of each rotor. V_A, V_B and V denote the volumes of each subsystem and of the total system respectively. We choose the coupling $J \sim \rho_s L^{d-2}$ appropriately to reflect the finite order-parameter stiffness of the system. It is straightforward to find the spectrum of the Hamiltonian (1.6) in the limit $\rho_s L^{d-1}/c \gg 1$. The entanglement properties of the ground state wave-function are precisely as described above.

Ideally, we would like to derive the effective Hamiltonian (1.6) starting from the $O(N)$ non-linear σ-model, which forms the full low-energy description of the system. At this point, we have not succeeded in doing so. Instead, we have performed a calculation of the entanglement entropy in the σ-model itself using the replica method. For a general N, taking the compact nature of the order parameter into account is non-trivial and our calculations are, strictly speaking, reliable only as long as the temperature T is much larger than the tower of states energy spacing $\Delta_{\text{tower}} \sim \frac{c^2}{\rho_s V}$. In this limit, the replica method calculations of entanglement entropy are in agreement with the results of the rotor model (1.6). Moreover, the rotor model indicates that the leading behavior of entanglement entropy is temperature independent for T much smaller than the spin-wave gap c/L, so our replica calculations remain correct down to zero temperature. We confirm this fact explicitly for the case $N = 2$, where replica calculations can be extended all the way to $T = 0$. For a general N, we also study the entanglement entropy in the presence of a symmetry breaking field $\vec{h}$, which lifts the degeneracy of the vacuum manifold, and find complete agreement between the rotor model and replica method results.

We would like to note that the presence of logarithmic corrections to the entanglement entropy of a Heisenberg antiferromagnet has been theoretically pointed out previously in Ref. 23. The authors of Ref. 23 have used the spin-wave (large S) expansion of the Heisen-

berg model. The effect of symmetry restoration in a finite volume has been mimicked by an application of a staggered magnetic field $h \sim 1/V^2$, suitably chosen to give a zero net staggered moment. The final step of the calculation of the entanglement entropy has been performed numerically and finite size scaling was used to extract the coefficient of the logarithmic divergence $b \approx 0.93$. This value is quite close to our exact result $b = 1$.[3] It was also found numerically that the coefficient of the corresponding divergence in the Renyi entropy S_n is practically independent of n in agreement with our exact conclusion.

The connection between the restoration of symmetry in a finite volume and the appearance of subleading logarithmic terms in the entanglement entropy has also been made in Ref. 13. Here the authors started with a mean-field Néel state of a Heisenberg antiferromagnet and averaged it over all orientations of the order parameter. The entanglement entropy of the resulting spin-singlet state was found to be $S = \log \mathcal{N} = d \log(L/a)$, where $\mathcal{N}$ is the total number of lattice sites (here and below we drop the constant piece in S). In contrast, our calculation, Eq. (1.5), gives for a Heisenberg antiferromagnet ($N = 3$), $\Delta S = (d-1) \log \left((\rho_s/c)^{1/(d-1)} L \right)$. The physical reason for the difference is that Ref. 13 does not take into account the existence of spin-waves, while as we have argued above, the presence of both the tower of states and spin-waves is crucial for the correction to entanglement entropy (1.4). The estimate of Ref. 13 for an antiferromagnet is physically similar to the exact result one obtains in the case of a free Bose gas where $S = \frac{1}{2} \log \mathcal{N} = \frac{d}{2} \log L/a$, with $\mathcal{N}$ - the total number of particles and a - the average interparticle distance. On the other hand, for an interacting Bose gas, *i.e.* a superfluid, our result, Eq. (1.5), with $N = 2$ gives $S = \frac{d-1}{2} \log \left((\rho_s/c)^{1/(d-1)} L \right)$. Thus, based on entanglement entropy we can distinguish a free Bose gas, where no Goldstone boson is present, from an interacting superfluid, which has a Goldstone mode.

This paper is organized as follows. Section II presents a calculation of entanglement properties of the toy rotor model (1.6). In section III we perform a calculation of entanglement entropy in the $O(N)$ non-linear σ-model using the replica method and show that the result is in complete agreement with that of the rotor model (certain details of the calculation are given in appendix A). Some concluding remarks are presented in section IV.

II.   ROTOR MODEL.

In this section, we calculate the Renyi entropy in the model (1.6), describing two quantum $O(N)$ rotors $\vec{n}_A$ and $\vec{n}_B$ of unit length. These rotors are taken to represent the average orientation of the order parameter in subsystems A and B. We choose the coupling J

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[3] We have repeated the calculation in Ref. 23 for system sizes up to 100×200 sites and found that b approaches unity (within a percent) as the system size is increased. We also note that the calculation in Ref. 23 can be reformulated using the Schwinger boson representation of spin operators. This yields an $SU(2)$ spin-symmetric mean-field ansatz wavefunction for the antiferromagnetic state that breaks the $SU(2)$ symmetry spontaneously in the thermodynamic limit.[24]

between the rotors in the following way. Recall that when the system is placed in a box of size L, the energy cost to twist the order parameter by an angle θ between the two sides of the box is related to the order-parameter stiffness ρ_s via,

$$\rho_s = \frac{1}{L^{d-2}} \frac{\partial^2 E}{\partial \theta^2}\bigg|_{\theta=0} \tag{2.1}$$

Thus, we take $J \sim \rho_s L^{d-2}$. Note that this value of J is approximate and should be understood in a scaling sense only.

To find the spectrum of the model (1.6) it is convenient to first work with the Lagrangian formulation,

$$L = \frac{\rho_s V_A (\partial_\tau \vec{n}_A)^2}{2c^2} + \frac{\rho_s V_B (\partial_\tau \vec{n}_B)^2}{2c^2} - J\vec{n}_A \cdot \vec{n}_B \tag{2.2}$$

Let us introduce the average and relative coordinates $\vec{n}$ and δn_α via,

$$
\begin{aligned}
n_A(\tau) &= \vec{n}\sqrt{1 - a^2(\delta n_\alpha)^2} + a\vec{e}_\alpha(\tau)\delta n_\alpha(\tau), \quad a = V_B/(V_A + V_B) \\
n_B(\tau) &= \vec{n}\sqrt{1 - b^2(\delta n_\alpha)^2} + b\vec{e}_\alpha(\tau)\delta n_\alpha(\tau), \quad b = -V_A/(V_A + V_B)
\end{aligned} \tag{2.3}
$$

Here, $\vec{n}^2 = 1$ and $\vec{e}_\alpha$ are $(N-1)$ unit vectors forming an orthonormal set together with $\vec{n}$. We expect the fluctuations of the average coordinate $\vec{n}$ to be parametrically slower than those of the relative coordinate δn. Moreover, we expect the fluctuations of δn to be small. Therefore, expanding the Lagrangian (2.2) to leading order in δn and in derivatives of $\vec{n}$ we obtain,

$$L \approx L_n + L_{\delta n} \tag{2.4}$$

$$L_n = \frac{\rho_s V}{2c^2}(\partial_\tau \vec{n})^2 \tag{2.5}$$

$$L_{\delta n} = \frac{\rho_s V_r}{2c^2}(\partial_\tau \delta n_\alpha)^2 + \frac{J}{2}(\delta n_\alpha)^2 \tag{2.6}$$

Here $V = V_A + V_B$ is the total volume and $V_r = V_A V_B/(V_A + V_B)$ is the reduced volume. We see that the Lagrangians of the average and relative coordinates decouple. The dynamics of the average coordinate $\vec{n}$ are governed by the quantum rotor Lagrangian (2.5); the associated Hamiltonian

$$H_n = \frac{c^2 \vec{L}_{\text{total}}^2}{2\rho_s V}, \quad \vec{L}_{\text{total}} = \vec{L}_A + \vec{L}_B, \tag{2.7}$$

is, indeed, the appropriate "tower of states" Hamiltonian describing the lowest energy excitations of a system with spontaneous symmetry breaking. On the other hand, the dynamics of the relative coordinate δn are governed by the Lagrangian (2.6) describing an $N-1$ dimensional harmonic oscillator. The frequency of this harmonic oscillator is given by

$$\omega = \sqrt{\frac{c^2 J}{\rho_s V_r}} \sim \frac{c}{L} \tag{2.8}$$

In the true physical system, the spectrum of the "relative motion" involves $N-1$ Goldstone modes with a dispersion $\omega = c|\vec{k}|$, where the momentum $\vec{k}$ is quantized in a finite geometry. Our rotor model (1.6) replaces this multi-mode spectrum with a single $N-1$ dimensional oscillator whose energy (2.8) is of order of the finite-size gap of the Goldstone modes. It turns out that such a replacement is sufficient for capturing the logarithmic contribution to the entanglement entropy (1.5).

The ground-state wave-function corresponding to the Lagrangian (2.4) is a product of a $\vec{L}_{\text{total}} = 0$ wave-function of the rotor Hamiltonian (2.7) and the ground state wave-function of the harmonic oscillator (2.6),

$$\psi(\vec{n}, \delta n) = \frac{1}{\sqrt{|S^{N-1}|}} \frac{1}{(\pi \xi^2)^{(N-1)/4}} \exp\left(-(\delta n_\alpha)^2/(2\xi^2)\right)$$

$$\approx \frac{1}{\sqrt{|S^{N-1}|}} \frac{1}{(\pi \xi^2)^{(N-1)/4}} \exp\left(-(\vec{n}_A - \vec{n}_B)^2/(2\xi^2)\right) \tag{2.9}$$

Here, $|S^{N-1}| = 2\pi^{N/2}/\Gamma(N/2)$ is the volume of a unit sphere S^{N-1} and

$$\xi = \left(\frac{c^2}{\rho_s V_r J}\right)^{1/4} \sim \left(\frac{c}{\rho_s L^{d-1}}\right)^{1/2} \ll 1 \tag{2.10}$$

Note that the condition $\xi \ll 1$ guarantees that the amplitude of the relative fluctuations is, indeed, small.

We proceed to compute the reduced density matrix ρ_A from the wave-function (2.9),

$$\rho_A(\vec{n}_A, \vec{n}'_A) = \int d\vec{n}_B \, \psi(\vec{n}_A, \vec{n}_B) \psi^*(\vec{n}'_A, \vec{n}_B) \tag{2.11}$$

The contributions to the integral over $\vec{n}_B$ above come from $|\vec{n}_A - \vec{n}_B|, |\vec{n}'_A - \vec{n}_B| \sim \xi \ll 1$. Therefore, $\rho_A(\vec{n}_A, \vec{n}'_A)$ is non-negligible only for $|\vec{n}_A - \vec{n}'_A| \ll 1$. We may change variables in Eq. (2.11) to $\delta \vec{n} = \vec{n}_A - \vec{n}_B$ and, to leading order, take both $\delta \vec{n}$ and $\vec{n}_A - \vec{n}'_A$ to lie in the tangent plane of $\vec{n}_A$. The integral over $\delta \vec{n}$ then becomes Gaussian and gives,

$$\rho_A(\vec{n}_A, \vec{n}'_A) = \frac{1}{|S^{N-1}|} \exp\left(-(\vec{n}_A - \vec{n}'_A)/(4\xi^2)\right) \tag{2.12}$$

Since $\vec{L}^2 = -\nabla^2$ is just the Laplacian on the sphere, we may use the heat kernel expansion

$$\langle \vec{n}|e^{-s\vec{L}^2}|\vec{n}'\rangle \rightarrow \frac{1}{(4\pi s)^{(N-1)/2}} \exp(-(\vec{n} - \vec{n}')^2/(4s)), \quad s \rightarrow 0 \tag{2.13}$$

Thus, as $\xi \ll 1$, we may write,

$$\rho_A \approx \frac{(4\pi \xi^2)^{(N-1)/2}}{|S^{N-1}|} e^{-\xi^2 \vec{L}_A^2} \tag{2.14}$$

Defining the entanglement Hamiltonian H_E as $\rho_A = \exp(-H_E)$, we obtain,

$$H_E = \xi^2 \vec{L}_A^2 + const \qquad (2.15)$$

Hence, the entanglement Hamiltonian has the same form as the physical tower of states Hamiltonian. We may write $H_E \sim H_A/T_E$ where $H_A = \frac{c^2 L_A^2}{2\rho_s V_A}$ is the Hamiltonian of the subsystem A and $T_E = \frac{c^2}{2\rho_s V_A \xi^2} \sim \frac{c}{L}$. Therefore, the reduced density matrix (2.14) describes a thermal state of the subsystem with the temperature of order of the Goldstone gap c/L.

We can now compute the Renyi entropy of the rotor model. From Eq. (2.14)

$$tr(\rho_A^n) \approx \frac{(4\pi\xi^2)^{n(N-1)/2}}{|S^{N-1}|^n} tr(e^{-n\xi^2 L_A^2}) \approx \frac{(4\pi\xi^2)^{(n-1)(N-1)/2}}{|S^{N-1}|^{n-1} n^{(N-1)/2}} \qquad (2.16)$$

where we've used Eq. (2.13) in the last step. Therefore,

$$S_n = -\frac{1}{n-1} \log tr(\rho_A^n) = \frac{N-1}{2}\left(\log\frac{1}{4\pi\xi^2} + \frac{\log n}{n-1}\right) + \log|S^{N-1}| \qquad (2.17)$$

$$\sim \frac{N-1}{2}\log\frac{\rho_s L^{d-1}}{c} \qquad (2.18)$$

Note that if we wish to interpret the result (2.18) in terms of the actual physical system rather than the toy rotor model, we must remember that Eq. (2.18) only captures the contribution of the universal low-lying part of the entanglement spectrum given by Eq. (2.15). There will also be a contribution from the non-universal, high-lying part of the entanglement spectrum, corresponding to short-range entanglement in the system. This contribution is expected to give a standard area law term in the Renyi entropy, so we identify the Renyi entropy of the rotor model, Eq. (2.18), with the ΔS contribution in Eq. (1.4). In addition, since the result (2.17) depends on ξ (J), and hence on the details of the spectrum of "relative fluctuations" (Goldstone modes), it is only logarithmically accurate. In particular, it does not capture the universal geometric constant $\gamma_{\rm ord}$ of Eq. (1.4).

It is easy to generalize the calculation above to the case when the system is at a finite temperature $T \ll \omega \sim c/L$. In this case, the Hamiltonian of the relative coordinate δn is still in its ground state, while the average coordinate $\vec{n}$ is in a thermal state. Thus, the total density matrix of the system is,

$$\rho(\vec{n}, \delta n; \vec{n}', \delta n') = \frac{1}{Z_{\vec{n}}}\langle\vec{n}|e^{-\frac{c^2\vec{L}^2}{2\rho_s VT}}|\vec{n}'\rangle \times \frac{1}{(\pi\xi^2)^{(N-1)/2}}\exp\left(-((\delta n_\alpha)^2 + (\delta n'_\alpha)^2)/(2\xi^2)\right)$$

$$\approx \frac{1}{Z_{\vec{n}}}\left\langle\vec{n}|e^{-\frac{c^2\vec{L}^2}{2\rho_s VT}}|\vec{n}'\right\rangle \times \frac{1}{(\pi\xi^2)^{(N-1)/2}}\exp\left(-((\vec{n}_A - \vec{n}_B)^2 + (\vec{n}'_A - \vec{n}'_B)^2)/(2\xi^2)\right)$$

$$(2.19)$$

where $Z_{\vec{n}} = tr(\exp(-\frac{c^2\vec{L}^2}{2\rho_s VT}))$ and $\vec{n}, \vec{n}'$ can be obtained from $\vec{n}_A, \vec{n}_B, \vec{n}'_A, \vec{n}'_B$ by inverting Eq. (2.3). To calculate the reduced density matrix we must set $\vec{n}_B = \vec{n}'_B$. The second factor

in Eq. (2.19) then implies that the density matrix vanishes for $|\vec{n}_A - \vec{n}_B|, |\vec{n}'_A - \vec{n}_B| \gg \xi$. On the other hand, the first factor in Eq. (2.19) varies on scale $|\vec{n} - \vec{n}'| \sim O(1)$ for $T \lesssim c^2/(\rho_s V)$ and on scale $|\vec{n} - \vec{n}'| \sim (c^2/(\rho_s V T))^{1/2}$ for $T \gg c^2/(\rho_s V)$. In either case this scale is much larger than ξ provided that $T \ll c/L$. Hence, we may set $\vec{n}_A = \vec{n}'_A = \vec{n}_B$ in the first factor,

$$\rho(\vec{n}_A, \vec{n}_B; \vec{n}'_A, \vec{n}_B) \approx \frac{1}{Z_{\vec{n}}} \langle \vec{n}_B | e^{-\frac{c^2 \vec{L}^2}{2\rho_s V T}} | \vec{n}_B \rangle \times \frac{1}{(\pi \xi^2)^{(N-1)/2}} \exp\left(-((\vec{n}_A - \vec{n}_B)^2 + (\vec{n}'_A - \vec{n}_B)^2)/(2\xi^2)\right)$$

$$= \frac{1}{|S^{N-1}|} \frac{1}{(\pi \xi^2)^{(N-1)/2}} \exp\left(-((\vec{n}_A - \vec{n}_B)^2 + (\vec{n}'_A - \vec{n}_B)^2)/(2\xi^2)\right) \qquad (2.20)$$

Note that the temperature dependence has disappeared in Eq. (2.20). Hence, for $T \ll c/L$ the Renyi entropy is to leading order given by our $T = 0$ result (2.18). One may understand this fact in the following way: by tracing over the subsystem B one puts the subsystem A at an effective temperature $T_E \sim c/L$, therefore, an initial physical temperature of the overall system $T \ll T_E$ can be neglected.

Finally, let us consider the situation when the $O(N)$ symmetry of the system is explicitly broken by an external field $\vec{h}$, which couples to the order parameters in the two subsystems as

$$\delta L = -M(V_A \vec{h} \cdot \vec{n}_A + V_B \vec{h} \cdot \vec{n}_B) \qquad (2.21)$$

where M is the expectation value of the microscopic order parameter (e.g. the staggered magnetization in the case when the system is an antiferromagnet). As before, changing variables to relative and average coordinates (2.3), we obtain,

$$\delta L \approx -MV\vec{h} \cdot \vec{n} \qquad (2.22)$$

Here, we assume that $MVh \ll J$ such that to leading order the external field does not modify the dynamics of the relative coordinate. At the same time, we will take MVh to be much larger than the tower of states spacing $c^2/(\rho_s V)$ so that the fluctuations of the average coordinate $\vec{n}$ about the direction of the external field are small. Then writing $\vec{n} = (\vec{\pi}, \sqrt{1 - \vec{\pi}^2})$ and expanding Eqs. (2.5), (2.22) in $\vec{\pi}$ we obtain,

$$L \approx L_n + L_{\delta n}$$
$$L_n \approx \frac{\rho_s V}{2c^2} (\partial_\tau \vec{\pi})^2 + \frac{MVh}{2} \vec{\pi}^2 \qquad (2.23)$$

with $L_{\delta n}$ still given by Eq. (2.6). The Lagrangian (2.23) describes an $N - 1$ dimensional harmonic oscillator with frequency m given by

$$m = \left(\frac{Mc^2 h}{\rho_s}\right)^{1/2} \qquad (2.24)$$

Note that the condition $c^2/(\rho_s V) \ll MVh \ll J$ ensures that m is much smaller than the frequency of relative motion ω (2.8), but much larger than the tower of states spacing

$c^2/(\rho_s V)$. Now, the ground state wave-function of the system is

$$
\begin{aligned}
\psi(\vec{\pi}, \delta n) &= \frac{1}{(\pi \xi \xi_h)^{(N-1)/2}} \exp\left(-\vec{\pi}^2/(2\xi_h^2)\right) \exp\left(-(\delta n_\alpha)^2/(2\xi^2)\right) \\
&\approx \frac{1}{(\pi \xi \xi_h)^{(N-1)/2}} \exp\left(-(1 - \hat{h} \cdot \vec{n})/\xi_h^2\right) \exp\left(-(\vec{n}_A - \vec{n}_B)^2/(2\xi^2)\right) \quad (2.25)
\end{aligned}
$$

where

$$
\xi_h = \left(\frac{c^2}{M\rho_s h V^2}\right)^{1/4} \quad (2.26)
$$

The conditions above ensure that $\xi \ll \xi_h \ll 1$. Computing the reduced density matrix, Eq. (2.11), we observe that as $\xi \ll \xi_h$ we may replace $\vec{n}$ in the first exponential in Eq. (2.25), by $\vec{n}_A$. Then integrating over $\vec{n}_B$,

$$
\begin{aligned}
\rho_A(\vec{n}_A, \vec{n}'_A) &\approx \frac{1}{(\pi \xi_h^2)^{(N-1)/2}} \exp\left(-(2 - \hat{h} \cdot (\vec{n}_A + \vec{n}'_A))/\xi_h^2\right) \exp\left(-(\vec{n}_A - \vec{n}'_A)^2/(4\xi^2)\right) \\
&\approx \frac{1}{(\pi \xi_h^2)^{(N-1)/2}} \exp\left(-(\vec{\pi}_A^2 + \vec{\pi}'^2_A)/(2\xi_h^2)\right) \exp\left(-(\vec{\pi}_A - \vec{\pi}'_A)^2/(4\xi^2)\right) \quad (2.27)
\end{aligned}
$$

where we've written $\vec{n}_A = (\vec{\pi}_A, \sqrt{1 - \vec{\pi}_A^2})$ in the last step. Now, consider a harmonic oscillator Hamiltonian

$$
H_E = \frac{\vec{p}_A^2}{2m_E} + \frac{1}{2} m_E \omega_E^2 \vec{\pi}_A^2 \quad (2.28)
$$

with $\vec{p}_A$ the momentum conjugate to $\vec{\pi}_A$. For $\omega_E \ll 1$,

$$
\begin{aligned}
\langle \vec{\pi}_A | \exp(-H_E) | \vec{\pi}'_A \rangle &\approx \left(\frac{m_E}{2\pi}\right)^{(N-1)/2} \exp\left(-m_E \omega_E^2 (\vec{\pi}_A + \vec{\pi}'_A)^2/8\right) \exp\left(-m_E(\vec{\pi}_A - \vec{\pi}'_A)^2/2\right) \\
&\approx \left(\frac{m_E}{2\pi}\right)^{(N-1)/2} \exp\left(-m_E \omega_E^2 (\vec{\pi}_A^2 + \vec{\pi}'^2_A)/4\right) \exp\left(-m_E(\vec{\pi}_A - \vec{\pi}'_A)^2/2\right)
\end{aligned}
$$
$$(2.29)$$

Therefore, we may write

$$
\rho_A \approx \omega_E^{N-1} e^{-H_E} \quad (2.30)
$$

with the identification,

$$
m_E = \frac{1}{2\xi^2}, \quad \omega_E = \frac{2\xi}{\xi_h} \quad (2.31)
$$

Note that $\omega_E \ll 1$, permitting the expansion (2.29). Thus, we see that in the present case the low-lying part of the entanglement spectrum is that of a harmonic oscillator. However, unlike in the absence of an external field, the reduced density matrix is not a thermal density matrix of the subsystem. Indeed, the amplitude of oscillations of $\vec{\pi}_A$ in the ground state of the entanglement Hamiltonian, (2.28), is $\xi_E = 1/\sqrt{m_E \omega_E} = \sqrt{\xi \xi_h}$, which is parametrically smaller than the physical amplitude of oscillations ξ_h. Hence, we cannot write the entanglement Hamiltonian in the form $H_E = H_A/T_E$.

It is now straight-forward to compute the Renyi entropy in the presence of an external field. From Eq. (2.30),

$$tr(\rho_A^n) = \omega_E^{n(N-1)} tr(e^{-nH_E}) \approx \left(\frac{\omega_E^{n-1}}{n} \right)^{N-1} \tag{2.32}$$

Hence,

$$S_n = -\frac{1}{n-1} \log tr\rho_A^n = (N-1) \left(\log \frac{\xi_h}{2\xi} + \frac{\log n}{n-1} \right) \tag{2.33}$$

$$\sim \frac{N-1}{2} \log \frac{c}{mL} \tag{2.34}$$

where we've expressed the final result in terms of the physical energy gap of the total system m. Note that Eq. (2.34) crosses over to our zero-field result (2.18) when m becomes of the order of the tower of states spacing $m \sim \frac{c^2}{\rho_s L^d}$. As before, Eq. (2.34) should be interpreted as the contribution ΔS in Eq. (1.4) to the Renyi entropy from the low-lying part of the entanglement spectrum. In addition, Eq. (2.33) is again by itself only logarithmically accurate since it involves the quantity ξ. However, we observe that ξ cancels out in the difference of zero and finite field Renyi entropies, Eqs. (2.17), (2.33),

$$S_n^{h=0} - S_n^h = \frac{N-1}{2} \left(\log \frac{1}{\pi\xi_h^2} - \frac{\log n}{n-1} \right) + \log |S^{N-1}| = \frac{N-1}{2} \left(\log \frac{\rho_s V m}{\pi c^2} - \frac{\log n}{n-1} \right) + \log |S^{N-1}| \tag{2.35}$$

As we see, the difference of Renyi entropies (2.35) depends only on the physical quantities: the system volume, the order-parameter stiffness and the finite-field mass gap and, hence, is fully universal, capturing not only the logarithmic dependence but also finite terms accurately. In section III we will reproduce Eq. (2.35) by a calculation in the $O(N)$ non-linear σ-model. This agreement makes us confident that the simple rotor model presented in this section accurately captures the universal entanglement properties of the system. Moreover, we will interpret the Renyi entropy of a system in a field as that of a free-bosonic scalar field theory, providing a way to calculate the universal constant $\gamma_{\rm ord}$ in Eq. (1.5) from a related constant in a free theory.

III.   REPLICA METHOD.

In this section, we calculate the entanglement entropy in the $O(N)$ non-linear σ-model using the replica method. The action of the theory is given by,

$$S = \frac{\rho_s}{2} \int d^dx d\tau \left(\frac{1}{c^2}(\partial_\tau \vec{n})^2 + (\nabla \vec{n})^2 \right) \tag{3.1}$$

with $\vec{n}$- an N-dimensional unit vector. We will set the spin-wave velocity $c = 1$ below and restore it in the final results.

We choose the system geometry to be a d-dimensional torus $\mathcal{T}^d$. For simplicity, we take all the directions of the torus to have length L. We choose the subsystem A to be the cylindrical region $\ell < \mathrm{x} < L$, with x - one of the directions of the torus. This is one of the geometries considered in the Monte-Carlo simulations of Ref. 13. Unless otherwise noted, we assume the ratio ℓ/L to be finite.

As is well known, the Renyi entropy S_n is given by

$$S_n = -\frac{1}{n-1} \log tr \rho_A^n = -\frac{1}{n-1} \log \frac{Z_n}{Z_1^n} \tag{3.2}$$

where Z_n is the partition function of the system on an n-sheeted Riemann surface.[1,2] For the geometry considered here this surface is given by $(\tau, \vec{x}) \in (0, n\beta) \times \mathcal{T}^d$ with $\beta = 1/T$ - the inverse temperature. The following identifications need to be made on this space:

$$\begin{aligned} (k\beta^+, \vec{x}) &\sim ((k+1)\beta^-, \vec{x}), \quad 0 < \mathrm{x} < \ell \\ (0^+, \vec{x}) &\sim (n\beta^-, \vec{x}), \quad \ell < \mathrm{x} < L \end{aligned} \tag{3.3}$$

Above, $0 \leq k \leq n-1$ is an integer. Below, we will be interested in the low temperature limit $\beta \gg L$.

To analyze the theory (3.1), we expand $\vec{n}$ about a given point on the unit sphere,

$$\vec{n} = (\vec{\pi}/\sqrt{\rho_s}, \sqrt{1 - \vec{\pi}^2/\rho_s}) \tag{3.4}$$

with $\vec{\pi}$ - an $N-1$ component real scalar field. At lowest order in energy expansion, the action (3.1) then becomes a free theory of the Goldstone modes $\vec{\pi}$,

$$S = \frac{1}{2} \int d^d x d\tau (\partial_\mu \vec{\pi})^2 \tag{3.5}$$

Here, μ runs over both space and time indices. In writing Eq. (3.5), we have lost the information about the compact nature of the order parameter manifold. We will partially restore the compactness later in the calculation. Thus,

$$\log Z_n = -\frac{N-1}{2} tr \log(-\partial^2)_n \tag{3.6}$$

where $(\partial^2)_n$ is the Laplacian on the n-sheeted Riemann surface. The calculation can be simplified by utilizing the translational invariance along the subsystem boundary,

$$\log Z_n = -\frac{N-1}{2} \sum_{\vec{k}_\parallel} tr \log((-\partial_\perp^2)_n + \vec{k}_\parallel^2) \tag{3.7}$$

with $\vec{k}_\parallel = \frac{2\pi}{L} \vec{m}$, $m \in \mathbb{Z}^{d-1}$ the momentum along the boundary. The operator $(\partial_\perp^2)_n$ now acts only in the two-dimensional (τ, x) space.

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So far, our calculation is identical to that in a free massless bosonic field theory. As long as the boundary conditions on the field are such that the energy gap is of order $1/L$ then one expects an entanglement entropy of the form (1.2). This was explicitly checked in Ref. 10, where twisted boundary conditions were assumed along the directions parallel to the subregion boundary, so that $|\vec{k}_\parallel| \geq \frac{\theta}{L}$, with θ -the twist angle. It is, therefore, clear that any violation of the scaling form (1.2) has to come from "quasi-zero" modes with eigenvalue λ^2 of $(-\partial^2)_n$ much smaller than $1/L^2$. This is in accord with our discussion in the introduction, which concluded that the violation of the scaling form (1.2) in a state with a spontaneously broken continuous symmetry is due to an emergence of the tower of states energy scale $\Delta_{\text{tower}} \sim \frac{1}{L^d}$, which is much smaller than the spin-wave gap $\frac{1}{L}$. Thus, we concentrate below on the contribution of quasi-zero modes to the entanglement entropy. As already noted, in a free theory quasi-zero modes can be eliminated by suitable boundary conditions, *e.g.* twisted boundary conditions on a torus or Dirichlet boundary conditions for open boundaries. However, for a physical system, in the absence of any external fields, the boundary conditions on $\vec{n}$ and, therefore, $\vec{\pi}$, will be periodic (non-twisted) along any periodic direction and von-Neumann for any open boundaries. Such boundary conditions admit a true zero mode with a constant $\vec{\pi}$, which will be discussed in more detail below. As we will now see, small temporal fluctuations on top of this zero mode give rise to quasi-zero modes.

Let us compute the spectrum of quasi-zero modes. Clearly, these must have $\vec{k}_\parallel = 0$. Let us now focus on the behavior of quasi-zero modes in the (τ, x) plane. Here, we must deal with the branch cuts at $\tau = k\beta$, $0 < \text{x} < \ell$. The behavior of the eigenmodes in the vicinity of the branch cuts is expected to be nontrivial. However, for $|\tau - k\beta| \gg L$, we expect the eigenfunctions ϕ of $-(\partial^2)_n$ to approach a linear superposition of plane wave states, $\phi \sim e^{i\omega\tau + 2\pi i m \text{x}/L}$, $m \in \mathbb{Z}$. For the eigenvalue $\lambda^2 = \omega^2 + (2\pi m/L)^2$ to be much smaller than $1/L^2$, we must choose $m = 0$. Therefore, we expect the following asymptotic behavior of quasi-zero modes with eigenvalue $\lambda^2 = \omega^2$,

$$\phi(\tau, \text{x}) = A_k^+ e^{i\omega(\tau - k\beta)} + A_k^- e^{-i\omega(\tau - k\beta)}, \quad \tau - k\beta \gg L, \ (k+1)\beta - \tau \gg L \qquad (3.8)$$

Without loss of generality, we take $\omega > 0$. To study the low temperature limit $\beta \gg L$ it is convenient to cut our Riemann surface into n separate sheets by defining

$$\phi_k(\tau, \text{x}) = \phi(k\beta + \tau, \text{x}) \qquad (3.9)$$

Here the variable k is defined modulo n. Each sheet has a branch-cut at $\tau = 0$, $0 < \text{x} < \ell$ and the sheets are glued together along these branch cuts,

$$\begin{aligned}
\phi_k(0^+, \text{x}) &= \phi_{k+1}(0^-, \text{x}), \quad 0 < \text{x} < \ell \\
\partial_\mu \phi_k(0^+, \text{x}) &= \partial_\mu \phi_{k+1}(0^-, \text{x}), \quad 0 < \text{x} < \ell
\end{aligned} \qquad (3.10)$$

At finite β, we should also glue the sheets via

$$\phi_k(\tau, \mathrm{x}) = \phi_{k+1}(\tau - \beta, \mathrm{x}) \tag{3.11}$$

However, since for $|\tau| \gg L$, ϕ_k approaches the plane wave states (3.8), we will take the τ coordinate in each sheet to run from $-\infty$ to ∞ and implement Eq. (3.11) via a boundary condition,

$$\begin{aligned}
\phi_k(\tau, \mathrm{x}) &= A_k^+ e^{i\omega\tau} + A_k^- e^{-i\omega\tau}, \quad \tau \to \infty \\
\phi_k(\tau, \mathrm{x}) &= A_{k-1}^+ e^{i\omega(\tau+\beta)} + A_{k-1}^- e^{-i\omega(\tau+\beta)}, \quad \tau \to -\infty
\end{aligned} \tag{3.12}$$

The corrections to this approximation are expected to be of order $e^{-\beta L}$. Thus, we must solve

$$-\partial^2 \phi_k = \omega^2 \phi_k \tag{3.13}$$

subject to the boundary conditions (3.10), (3.12).

We may further simplify Eq. (3.13) in the quasi-zero mode limit $\omega \ll 1/L$. Indeed, in the vicinity of the branch points, we expect ϕ to vary on the length-scale L. Thus, the typical contributions to the left hand side of Eq. (3.13) are of order $1/L^2$ and we may set the right hand side of Eq. (3.13) to zero,

$$-\partial^2 \phi_k = 0 \tag{3.14}$$

We expect the approximation (3.14) to hold as long a $|\tau| \ll 1/\omega$. On the other hand, once $|\tau| \gg L$ the asymptotic forms (3.12) become valid, so we should solve Eq. (3.14) subject to the boundary conditions,

$$\begin{aligned}
\phi_k(\tau, \mathrm{x}) &= (A_k^+ + A_k^-) + i(A_k^+ - A_k^-)\omega\tau, \quad \tau \to \infty \\
\phi_k(\tau, \mathrm{x}) &= A_{k-1}^+ e^{i\omega\beta} + A_{k-1}^- e^{-i\omega\beta} + i(A_{k-1}^+ e^{i\omega\beta} - A_{k-1}^- e^{-i\omega\beta})\omega\tau, \quad \tau \to -\infty
\end{aligned} \tag{3.15}$$

together with the gluing condition (3.10).

We can solve the Laplace equation (3.14) using conformal mapping. Let us define $z = \mathrm{x} + i\tau$ on each sheet of the Riemann surface. Topologically, each sheet is a cylinder due to the periodicity of x. We begin by mapping each cylinder to a complex plane using $w = e^{2\pi i z/L}$. This maps the branch cut at $\tau = 0$, $0 < \mathrm{x} < \ell$ to a segment of a unit circle with angle $0 < \theta < e^{2\pi i \ell/L}$. Moreover, $\tau = \infty$ is mapped to the origin and $\tau = -\infty$ to the point at infinity. Next, we apply the following map to each sheet,

$$\zeta = e^{-\pi i \ell/L} \frac{w - e^{2\pi i \ell/L}}{1 - w} \tag{3.16}$$

This maps the branch cut into the positive x-axis. Moreover, $\tau = \pm\infty$ is now mapped to $-e^{\pm\pi i \ell/L}$. Finally, we map the n-sheets into a single sheet using $s = \zeta^{1/n}$. More precisely, for ζ on the k-th sheet with $\zeta = re^{i\theta}$, $r > 0$, $0 < \theta < 2\pi$, we define $\zeta^{1/n} = r^{1/n} e^{i(\theta + 2\pi k)/n}$.

With this definition and the gluing (3.10), there are no branch cuts in the s plane. Note that the points $\tau = \pm\infty$ on the kth sheet now map into $s_k^\pm = e^{\pi i(2k+1\pm\ell/L)/n}$. The boundary conditions (3.15) imply logarithmic divergences for $s \to s_k^\pm$. Note that these are the only singularities that can occur in the s-plane, including the point $s = \infty$. Indeed, $s = \infty$ is the image of the branch point $\tau = 0$, $x = 0$ at which we expect no singularity. Therefore,

$$\phi(s) = \sum_k \left(c_k^+ \log|s - s_k^+| + c_k^- \log|s - s_k^-| \right) + C, \tag{3.17}$$

Note that the absence of a singularity at $s = \infty$ implies,

$$\sum_k \left(c_k^+ + c_k^- \right) = 0 \tag{3.18}$$

The solution (3.17) has the following asymptotic behavior as $\tau \to \pm\infty$ on kth sheet,

$$\phi_k(\tau, x) \overset{\tau \to \infty}{\to} -\frac{2\pi\tau}{L} c_k^+ + \log\left(\frac{2}{n}\sin(\pi\ell/L)\right) c_k^+ + \sum_{l\neq k} c_l^+ \log|s_k^+ - s_l^+| + \sum_l c_l^- \log|s_k^+ - s_l^-| + C,$$

$$\phi_k(\tau, x) \overset{\tau \to -\infty}{\to} \frac{2\pi\tau}{L} c_k^- + \log\left(\frac{2}{n}\sin(\pi\ell/L)\right) c_k^- + \sum_l c_l^+ \log|s_k^- - s_l^+| + \sum_{l\neq k} c_l^- \log|s_k^- - s_l^-| + C$$

$$\tag{3.19}$$

Imposing the boundary conditions (3.15) we obtain,

$$c_k^+ = \frac{-i\omega L}{2\pi}(A_k^+ - A_k^-) \tag{3.20}$$

$$c_k^- = \frac{i\omega L}{2\pi}(A_{k-1}^+ e^{i\omega\beta} - A_{k-1}^- e^{-i\omega\beta}) \tag{3.21}$$

and

$$A_k^+ + A_k^- - C = \log\left(\frac{2}{n}\sin(\pi\ell/L)\right) c_k^+ + \sum_{l\neq k} c_l^+ \log|s_k^+ - s_l^+| + \sum_l c_l^- \log|s_k^+ - s_l^-|$$

$$A_{k-1}^+ e^{i\omega\beta} + A_{k-1}^- e^{-i\omega\beta} - C = \log\left(\frac{2}{n}\sin(\pi\ell/L)\right) c_k^- + \sum_l c_l^+ \log|s_k^- - s_l^+| + \sum_{l\neq k} c_l^- \log|s_k^- - s_l^-|$$

$$\tag{3.22}$$

Note that from (3.21) the right-hand side of Eq. (3.22) is suppressed compared to the left-

hand side by a factor of ωL, and at leading order may be dropped. Thus, we must solve

$$A_k^+ + A_k^- = C \tag{3.23}$$

$$A_{k-1}^+ e^{i\omega\beta} + A_{k-1}^- e^{-i\omega\beta} = C \tag{3.24}$$

$$(1 - e^{i\omega\beta}) \sum_k A_k^+ - (1 - e^{-i\omega\beta}) \sum_k A_k^- = 0 \tag{3.25}$$

Here, Eq. (3.25) comes from the condition (3.18). To further simplify the above equations, we observe that our original problem is symmetric under the time translation $\tau \to \tau + \beta$, which generates a cyclic permutation of the n sheets. In the s plane it maps $s \to e^{2\pi i/n} s$. We choose ϕ to be an eigenstate of this symmetry with eigenvalue $e^{2\pi i p/n}$, which implies,

$$A_k^\pm = A^\pm e^{2\pi i p k/n} \tag{3.26}$$

For $p \neq 0$, the symmetry also constrains $C = 0$. Moreover, in this case Eq. (3.25) is trivially satisfied. Then, solving Eqs. (3.23),(3.24) we obtain $A_+ = -A_-$ and the condition $\sin(\omega\beta) = 0$, *i.e.*

$$\omega = \frac{\pi m}{\beta}, \; m \in \mathbb{N}, \quad p \neq 0 \tag{3.27}$$

Thus, for "momentum" $p \neq 0$, one mode is present for each m in Eq. (3.27). On the other hand, for $p = 0$, we obtain $C = A_+ + A_-$ and $e^{i\omega\beta} = 1$, *i.e.*,

$$\omega = \frac{2\pi m}{\beta}, \; m \in \mathbb{N}, \quad p = 0 \tag{3.28}$$

In this case, A_+ and A_- are independent, so two modes are present for each m in Eq. (3.28). The explicit form of the corresponding eigenstates is,

$$\phi = A^+ + A^- - \frac{i\omega L}{2\pi}(A^+ - A^-) \sum_k \log \left| \frac{s - s_k^+}{s - s_k^-} \right| = A^+ + A^- + i\omega\tau(A^+ - A^-) \tag{3.29}$$

We have inverted the conformal mapping in the last step. Upon continuing Eq. (3.29) from $|\omega\tau| \ll 1$ to the full range of τ we obtain,

$$\phi = A^+ e^{i\omega\tau} + A^- e^{-i\omega\tau} \tag{3.30}$$

For ω satisfying (3.28), Eq. (3.30) is actually the exact eigenstate of $-\partial^2$ on the n-sheeted surface. In contrast, the solutions with $p \neq 0$ that we've constructed are not exact. The leading correction to these eigenstates can be calculated by keeping the terms on the right-hand side of Eqs. (3.22). This gives an $O(L/\beta)$ corrections to the eigenvalues (3.27). We have verified that these corrections do not modify any of the leading order results below for the Renyi entropy.

We are now ready to compute the contribution of zero and quasi-zero modes to the

partition function.

$$\log Z_n = \log S_0 - \frac{N-1}{2} tr' \log(-\partial^2)_n \tag{3.31}$$

As already mentioned, in addition to the quasi-zero modes discussed above, $-\partial^2$ has one true zero mode on the n-sheeted Riemann surface, which is simply given by constant ϕ. S_0 in Eq. (3.31) is the zero-mode contribution and tr' is the trace with the zero-mode removed. The trace in Eq. (3.31) requires regularization. We will use Pauli-Villars regularization here by subtracting the free energy on an n-sheeted Riemann surface of a free boson field with mass Λ, which acts as a cut-off. Since we are considering only modes with eigenvalue $\lambda^2 \ll 1/L^2$, the cut-off Λ should be taken to be $\Lambda \sim 1/L$. Thus,

$$\log Z_n = \log S_0 - \frac{N-1}{2} tr' \log(-\partial^2)_n + \frac{N-1}{2} tr \log((-\partial^2)_n + \Lambda^2) \tag{3.32}$$

Note that the regulator trace includes the zero-mode of $-\partial^2$. Let us begin by calculating the zero mode contribution S_0. Physically, the presence of the zero-mode indicates the degeneracy of the vacuum manifold. Indeed, we have expanded our field $\vec{n}$ around a fixed direction. However, in reality, we should integrate over all directions of $\vec{n}$. Thus, S_0 should be converted into an integral over the order parameter manifold. Locally the manifold can be parameterized by

$$\vec{\pi}(x) = \vec{\pi}_0 \tag{3.33}$$

and

$$S_0 = \int d^{N-1}\vec{\pi}_0 \sqrt{det g} \tag{3.34}$$

with the metric

$$g_{ij} = \frac{1}{2\pi} \int dx \frac{\partial \vec{\pi}(x)}{\partial \pi_0^i} \cdot \frac{\partial \vec{\pi}(x)}{\partial \pi_0^j} = \frac{n\beta L^d}{2\pi} \delta_{ij} \tag{3.35}$$

Thus,

$$S_0 = \left(\frac{n\beta L^d}{2\pi}\right)^{(N-1)/2} \int d^{N-1}\vec{\pi}_0 \approx \left(\frac{n\beta \rho_s L^d}{2\pi}\right)^{(N-1)/2} \int d\vec{n} = \left(\frac{n\beta \rho_s L^d}{2\pi}\right)^{(N-1)/2} |S^{N-1}| \tag{3.36}$$

where we have gone from a local integral over $\vec{\pi}_0$ to a global integral over the order parameter orientation $\vec{n}$. Here, $|S^{N-1}| = 2\pi^{N/2}/\Gamma(N/2)$ is the volume of an $N-1$ dimensional unit sphere.

Next, consider the contribution of quasi-zero modes,

$$
\begin{aligned}
\log Z_n^{qz} &= -\frac{N-1}{2}(tr'\log(-\partial_n^2) - tr'\log((-\partial^2)_n + \Lambda^2)) \\
&= -\frac{N-1}{2}\Bigg[(n-1)\sum_{m=1}^{\infty}\left(\log(\pi m/\beta)^2 - \log\left((\pi m/\beta)^2 + \Lambda^2\right)\right) \\
&\quad + 2\sum_{m=1}^{\infty}\left(\log(2\pi m/\beta)^2 - \log\left((2\pi m/\beta)^2 + \Lambda^2\right)\right)\Bigg]
\end{aligned}
\tag{3.37}
$$

The first sum in the square brackets gives the contribution of modes (3.27) with $p \neq 0$, while the second sum gives the contribution of modes (3.28) with $p = 0$. Using the standard relation,

$$
-\frac{1}{2}\sum_{m=-\infty}^{\infty}\left(\log((2\pi m/\beta)^2 + \omega^2) - \log((2\pi m/\beta)^2 + \omega'^2)\right) = \log Z_{ho}(\omega,\beta) - \log Z_{ho}(\omega',\beta)
\tag{3.38}
$$

where $Z_{ho}(\omega,\beta)$ is the partition function of a harmonic oscillator with frequency ω at inverse temperature β,

$$
Z_{ho} = \frac{1}{2\sinh(\beta\omega/2)}
\tag{3.39}
$$

we obtain in the limit $\beta\Lambda \gg 1$,

$$
\log Z_n^{qz} = -\frac{N-1}{2}\left(-n\beta\Lambda + (n+1)\log(\beta\Lambda) + (n-1)\log 2\right)
\tag{3.40}
$$

Combining (3.40) with (3.36) and taking the contribution of the zero mode to the regulator trace in Eq. (3.32) into account, we obtain

$$
\log Z_n = \frac{N-1}{2}\left[n\beta\Lambda + \log(n\rho_s L^d/(2\pi\beta)) - (n-1)\log(2\beta\Lambda)\right] + \log|S^{N-1}|
\tag{3.41}
$$

which gives the Renyi entropy,

$$
\begin{aligned}
\Delta S_n &= -\frac{1}{n-1}\log\frac{Z_n}{Z^n} = \frac{N-1}{2}\left[\log(\rho_s L^d \Lambda/\pi) - \frac{\log n}{n-1}\right] + \log|S^{N-1}| \tag{3.42} \\
&\approx \frac{N-1}{2}\log\frac{\rho_s L^{d-1}}{c} \tag{3.43}
\end{aligned}
$$

Note that we have set the cut-off $\Lambda \sim 1/L$ in the last step, so that Eq. (3.43) is valid only to logarithmic accuracy and we have dropped the constant terms in Eq. (3.42). We will come back to give an interpretation of these terms below. We have also restored the spin-wave velocity c in Eq. (3.43).

A few comments are in order here. First of all, the result (3.43) takes into account only the contribution of zero and quasi-zero modes to the entanglement entropy. As argued in the

introduction, the higher energy modes are expected to give a CFT like contribution (1.2). Second, the result (3.43) agrees with the expression (2.18) obtained in the toy rotor model of section II. Third, our calculation above is strictly justified only for temperature much greater than the tower of states energy spacing, $T \gg \Delta_{\text{tower}} \sim \frac{1}{\rho_s L^d}$. This can be seen from studying the physical free energy $F(T) = -\frac{1}{\beta} \log Z_1$. Observe that Eq. (3.41) gives,

$$F(T) - F(0) = -T \left(\frac{N-1}{2} \log \frac{\rho_s L^d T}{2\pi} + \log |S^{N-1}| \right) \tag{3.44}$$

This is in agreement with the free energy of the rotor Hamiltonian (1.3) for $T \gg \Delta_{\text{tower}}$, but not for $T \ll \Delta_{\text{tower}}$, where the physical $F(T) - F(0)$ is exponentially suppressed. The technical reason for the disagreement between our path-integral treatment and the Hamiltonian calculation is that the former takes the compactness of the order parameter into account only in the treatment of zero modes, whereas the finite frequency modes are still described by a non-compact free boson theory. This approximation is reliable for $T \gg \Delta_{\text{tower}}$ where the fluctuations of $\vec{n}$ corresponding to the finite frequency modes are small, but breaks down for T below Δ_{tower} where these fluctuations are large. However, we observe that while our free energy has an unphysical temperature dependence for $T \ll \Delta_{\text{tower}}$, this temperature dependence disappears in the expression for the Renyi entropy (3.43). Thus, one may hope that the result (3.43) is correct not only for $\Delta_{\text{tower}} \ll T \ll c/L$, but actually for all $T \ll c/L$. The calculations of section II in the toy rotor model support this hypothesis. In appendix A we confirm this hypothesis by a replica method calculation for the special case $N = 2$, where the compactness of the order parameter manifold can be taken into account exactly for all temperatures.

Let us also analyze the entanglement entropy in the presence of a small symmetry breaking field $\vec{h}$ coupled to the order parameter as,

$$\delta S = -M \int d^d x d\tau \, \vec{h} \cdot \vec{n} \tag{3.45}$$

Here, M is the expectation value of the order parameter. In the case of a Heisenberg antiferromagnet the symmetry breaking field corresponds to a staggered magnetic field. Using the representation (3.4) and expanding the action to leading order in $\vec{\pi}$

$$S = \frac{1}{2} \int d^d x d\tau \left((\partial_\mu \vec{\pi})^2 + m^2 \vec{\pi}^2 \right) \tag{3.46}$$

with the mass $m^2 = hM/\rho_s$. As in section II, we will consider the regime when the field induced Goldstone mass is much smaller than the spin-wave gap $1/L$, but much larger than the tower of states gap $\Delta_{\text{tower}} \sim 1/(\rho_s L^d)$. In this regime, the spectrum of spin-waves with finite momentum is unaffected by the symmetry breaking field. On the other hand, the zero-momentum fluctuations of the order parameter about the staggered field are now small and described by a harmonic oscillator with frequency m. Thus, in this field range, one

may ignore the compactness of the order parameter manifold and work with the free theory (3.46). The regularized partition function on the n-sheeted Riemann surface now becomes

$$\log Z_n = -\frac{N-1}{2} \left(tr \log((-\partial^2)_n + m^2) - tr \log((-\partial^2)_n + \Lambda^2) \right) \tag{3.47}$$

We again consider only the contribution of zero and quasi-zero modes of $(-\partial^2)_n$ to the partition function (3.47). Using the spectrum (3.27), (3.28)

$$
\begin{aligned}
\log Z_n = {} & -\frac{N-1}{2} \left[(n-1) \sum_{m=1}^{\infty} \left(\log((\pi m/\beta)^2 + m^2) - \log\left((\pi m/\beta)^2 + \Lambda^2\right) \right) \right. \\
& \left. + 2 \sum_{m=1}^{\infty} \left(\log((2\pi m/\beta)^2 + m^2) - \log\left((2\pi m/\beta)^2 + \Lambda^2\right) \right) + \log(m^2/\Lambda^2) \right]
\end{aligned}
\tag{3.48}
$$

where the last term in the second line of Eq. (3.48) denotes the contribution of the zero mode of $(-\partial^2)_n$. Performing the sum and taking the limit $T \to 0$,

$$\log Z_n = \frac{N-1}{2} \left(n\beta(\Lambda - m) - (n-1)\log \Lambda/m \right) \tag{3.49}$$

which gives the Renyi entropy,

$$\Delta S_n = \frac{N-1}{2} \log \Lambda/m \tag{3.50}$$

$$\approx \frac{N-1}{2} \log \frac{1}{mL} \tag{3.51}$$

$$= \frac{N-1}{4} \log \frac{\rho_s}{hML^2} \tag{3.52}$$

Here, we have set $\Lambda \sim 1/L$ in Eq. (3.51). Observe that Eq. (3.52) is in agreement with the result (2.34) of the rotor model. Also note that Eq. (3.51) implies that the Renyi entropy of a free theory diverges logarithmically in the massless limit $m \to 0$. In the case of a system with a spontaneously broken symmetry, the calculations above are, however, valid only as long as m is much larger than the tower of states gap $\Delta_{\text{tower}} \sim \frac{1}{\rho_s L^d}$, and at $m \sim \Delta_{\text{tower}}$, Eq. (3.52) crosses over to our previous result (3.43). Since the present calculation is performed at $T = 0$ it supports the hypothesis that Eq. (3.43) remains correct down to $T = 0$.

Let us discuss the connection between the entanglement entropy of a free theory and a state with a spontaneously broken continuous symmetry in more detail. For a free single component real scalar field with mass m, we expect the Renyi entropy to behave as,

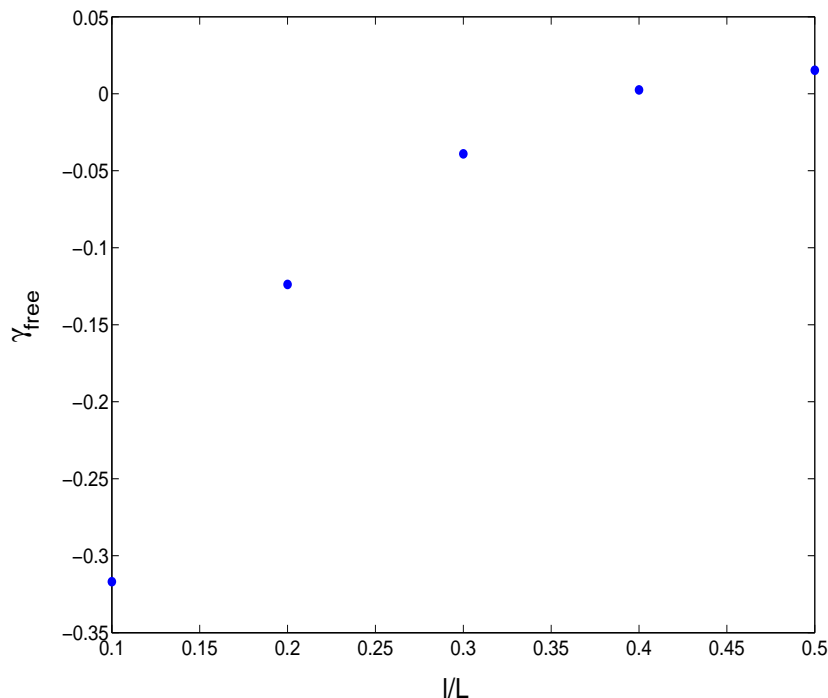$$S_n = C\frac{\mathcal{A}}{a^{d-1}} + \Delta S_{\text{free}}(mL) \tag{3.53}$$

FIG. 1: The universal constant contribution γ_{free}, Eq. (3.54), to the Renyi entropy S_2 of a free massive scalar in two dimensions as a function of the ratio ℓ/L. The total system is a torus with dimensions $L \times L$, while the subsystem is a cylinder of size $\ell \times L$. We take the mass $m = 10^{-9}$ and study systems with size L up to 100 sites, so as to fit the universal part of the entanglement entropy to the scaling form in Eq. (3.54). γ_{free} is related to the universal constant γ_{ord} in the ordered phase via Eq. (3.55).

with ΔS_{free} a universal geometry dependent function.[4] Here we assume all the usual conditions about smoothness/flatness of the subsystem boundary. Eq. (3.51) demonstrates that for $mL \to 0$, ΔS_{free} takes the form,

$$\Delta S_{\text{free}} = \frac{1}{2} \log \frac{1}{mL} + \gamma_{\text{free}} \qquad (3.54)$$

The constant term γ_{free} receives a contribution from modes with energy $\omega \sim 1/L$ and hence cannot be extracted from the present calculation. In fact, given our regularization of the trace (3.47), Eq. (3.50) is precisely the difference between the Renyi entropies of a theory with mass $m \ll 1/L$ and a theory with mass $\Lambda \ll 1/L$, so the constant γ_{free} cancels out in the result. Similarly, with the regularization (3.32), our result (3.42) is the difference between the Renyi entropy of a system with spontaneously broken continuous symmetry in

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[4] In dimension $d = 3$, the area law coefficient receives an additional singular contribution, resulting in a further correction to the entanglement entropy $\delta S \sim m^2 \mathcal{A} \log(ma)$.[7] However, this correction is negligible in the regime $mL \ll 1$, which is of main interest to the present work.

a zero external field and the Renyi entropy of a system in a field, which induces a mass gap Λ satisfying $\Delta_{\text{tower}} \ll \Lambda \ll 1/L$. This is precisely the quantity we considered in Eq. (2.35) of section II, which fully agrees with our replica method result (3.42). Moreover, since the system in a field is just described by an $N-1$ component free scalar field, we may combine Eqs. (1.5), (3.54), (3.42) to obtain,

$$\gamma_{\text{ord}} = (N-1)\left(\gamma_{\text{free}} - \frac{1}{2}\log\pi - \frac{\log n}{2(n-1)}\right) + \log|S^{N-1}| \qquad (3.55)$$

Thus, the geometric constant γ_{ord} in the ordered state is directly expressed in terms of γ_{free}. We have verified the scaling form, Eq. (3.54), for a lattice regularized free massive bosonic theory in $d=2$ using the correlation matrix method of Ref. 14, and computed the γ_{free} term in S_2 as a function of the ratio ℓ/L. The results are shown in Fig. 1. We take the mass $m = 10^{-9}$, the total system as a torus of size $L \times L$ and the subsystem as an $\ell \times L$ cylinder. We study systems with L up to 100 sites to extract γ_{free}.

Our result (3.54) may appear to contradict existing calculations in a free bosonic theory (see *e.g.* the review Ref. 26), which find the entanglement entropy to remain finite in the massless limit in dimension $d > 1$, while in $d = 1$ a very weak divergence $\Delta S \sim \frac{1}{2}\log\log 1/(m\ell)$ is observed.[27] However, the geometry studied in Refs. 26,27 involves a subsystem of finite size ℓ embedded in a system whose size L is taken to infinity from the outset. The limits $m \to 0$ and $L \to \infty$ do not commute; our result (3.54) corresponds to taking $m \to 0$ while holding the ratio ℓ/L-fixed.

IV. CONCLUSION

In this paper we have demonstrated the presence of logarithmic corrections to the entanglement entropy in systems with spontaneous breaking of continuous symmetry. Such corrections have been recently observed in Monte-Carlo simulations.[13] Our result, Eq. (1.5) gives the coefficient of the logarithmic divergence to be $b = 1$ in the case of a Heisenberg antiferromagnet ($N = 3$) in two spatial dimensions. Presently, the Monte-Carlo simulations give $b = 0.74 \pm 0.02$.[13] We believe that the difference between our exact result and the Monte-Carlo comes from the difficulty in extracting a subleading correction to the Renyi entropy in systems of relatively small size (up to 20×20 lattice sites) studied in Ref. 13. In addition to the coefficient of the logarithmic correction, we also give an expression, Eq. (3.55), for the finite geometric constant γ_{ord} in Eq. (1.5) in terms of a related constant γ_{free} in a free massive bosonic theory, see Eq. (3.54). We evaluate the constant γ_{free} as a function of the subsystem to total system size ratio ℓ/L using the correlation matrix method of Ref. 14 (Fig. 1). We would like to point out that the behavior of the geometric constant γ_{ord} has very recently been investigated by Monte-Carlo simulations of the Heisenberg model[25] - the shape of $\gamma_{\text{free}}(\ell/L)$ (and hence of $\gamma_{\text{ord}}(\ell/L)$) in Fig. 1 appears to be at least in qualitative agreement with the Monte-Carlo results of Ref. 25.

A curious byproduct of our work is that the Renyi entropy of a free bosonic theory diverges

as the boson mass $m \to 0$, see Eq. (3.54). This is a consequence of the shift symmetry of the free massless theory $\phi \to \phi + \text{const.}$ (here we assume that the boundary conditions on the field do not break this symmetry). If the field ϕ is non-compact, the symmetry group is $\mathbb{R}$. When one computes an expectation value of an operator O which is invariant under the above symmetry, $\langle O \rangle = \int D\phi\, O e^{-S[\phi]} / \int D\phi\, e^{-S[\phi]}$, the infinite factor of the group volume cancels between the numerator and denominator. However, when one computes the Renyi entropy, $S_n = -\frac{1}{n-1} \log(Z_n/Z^n)$, the group volume no longer cancels - this is, essentially, the origin of the $m \to 0$ divergence (3.54). If the field ϕ is compact, then the $m \to 0$ divergence is cut-off by the tower of states energy spacing, giving our main result (1.5). Note that the divergence of the Renyi entropy in the free non-compact scalar theory (3.54) is present in any dimension, including $d = 1$. Thus, the famous result for the entanglement entropy of a free boson in one dimension $S = \frac{1}{3} \log L/a$ implicitly assumes compactness of the boson field. The compactification radius is secretly hidden in the short-distance cut-off a. The $m \to 0$ divergence of Eq. (3.54) naively appears to contradict previous calculations[26], which have found the entanglement entropy of a free non-compact bosonic theory to remain finite in the massless limit if the dimension $d > 1$ (a weak $\log\log(1/(m\ell))$ divergence was found in $d = 1$[27]). However, these calculations have taken the total system size L to infinity first, and then $m \to 0$ while keeping the subsystem size ℓ fixed. This order of limits results in the entanglement entropy having the same behavior as in a CFT. On the other hand, in Eq. (3.54) we are considering a different order of limits, where ℓ/L is kept fixed and m is taken to zero. Note that in the case of a state with spontaneous symmetry breaking and no external symmetry breaking field we expect that taking $L \to \infty$ with ℓ fixed simply results in a replacement of L by ℓ in our result (1.5).

The present paper has mainly focused on the case when the subsystem boundary is smooth in $d = 2$ and flat in $d = 3$. When this condition is violated, the entanglement entropy is known to acquire additional logarithmic contributions. For instance, in the case $d = 2$, when the subsystem boundary has corners, the entanglement entropy of a CFT receives a contribution

$$\Delta S_{\text{corn}} = \sum_i b_{\text{corn}}(\varphi_i) \log L/a \tag{4.1}$$

Here the sum is over the corners of the boundary and the coefficient $b_{\text{corn}}(\varphi_i)$ depends on the corner angle φ_i. Such a corner contribution has been found in a free massive bosonic theory,[8]

$$\Delta S_{\text{corn}} = \sum_i b_{\text{corn}}(\varphi_i) \log \frac{1}{ma}, \quad m \gg 1/L \tag{4.2}$$

where an important condition $m \gg 1/L$ is assumed. For instance, for a 90° degree angle one obtains, $b_{\text{corn},n=1}(\pi/2) \approx -0.012$ for the entanglement entropy proper and $b_{\text{corn},n=2}(\pi/2) \approx -0.0062$ for the Renyi entropy with $n = 2$. In a CFT perturbed by a relevant operator which produces a mass gap m, the expression (4.2) crosses over to Eq. (4.1) once $m \ll 1/L$. However, as we discussed above, in a free bosonic theory with $m \to 0$ there is an additional

contribution to the entanglement entropy (3.54), so we expect to logarithmic accuracy[5]

$$S_n = C\frac{\mathcal{A}}{a^{d-1}} + \sum_i b_{\mathrm{corn}}(\varphi_i)\log\frac{L}{a} + \frac{1}{2}\log\frac{1}{mL}, \quad m \ll \frac{1}{L} \tag{4.3}$$

Note that the corner coefficients $b_{\mathrm{corn}}(\varphi)$ are controlled by ultra-violet physics and so are the same in Eqs. (4.2) and (4.3). The reason for the simple addition of the contributions (4.1) and (3.54) to the entanglement entropy is the separation of energy scales from which these originate; the contribution (4.1) comes the energy range $1/L \ll \omega \ll 1/a$ and the contribution (3.54) from $m \ll \omega \ll 1/L$. As before, for a system with spontaneous breaking of continuous symmetry the $m \to 0$ divergence of Eq. (4.3) is cut-off once m becomes of the order of tower of states energy spacing $\Delta_{\mathrm{tower}} \sim \frac{1}{\rho_s L^d}$ so

$$S_n = C\frac{\mathcal{A}}{a^{d-1}} + (N-1)\sum_i b_{\mathrm{corn}}(\varphi_i)\log\frac{L}{a} + \frac{N-1}{2}\log(\rho_s L^{d-1}/c) \tag{4.4}$$

In addition to the Renyi entropy, in the present paper we have also studied the entanglement spectrum of systems with spontaneous breaking of continuous symmetry. We have shown that the "low-energy" part of the entanglement spectrum takes the same "tower of states" form as the physical spectrum of the system. The level spacing of this universal part of the entanglement spectrum scales with the system size as $c/(\rho_s L^{d-1})$. We hope that this finding will be verified by exact diagonalization on small systems or DMRG calculations on multi-leg ladders.

Acknowledgments

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[5] As above, for Eq. (4.3) to be valid, it is important that the $m \to 0$ limit is taken with the ratio ℓ/L - fixed; if the total system size L is taken to infinity first and then $m \to 0$ at fixed ℓ then the CFT expression (4.1) is, indeed, obtained as found in Ref. 8.

Appendix A: Explicit calculation of Renyi entropy for $N = 2$ at $T = 0$

In this appendix we show that Eq. (3.43), indeed, remains correct at $T = 0$ by an explicit calculation in the case $N = 2$. Here, we may use the angular representation

$$\vec{n} = (\cos\phi, \sin\phi) \tag{A1}$$

with the action,

$$S = \frac{\rho_s}{2} \int d^d x d\tau (\partial_\mu \phi)^2 \tag{A2}$$

The compactness of the order parameter manifold is now expressed through the identification $\phi \sim \phi + 2\pi$.

Since the theory (A2) is free, we can simply repeat our previous calculation of the entanglement entropy. The only additional complication is that we should restore the periodicity of the variable ϕ. This is accomplished by summing over all winding number configurations of ϕ. More precisely, each field configuration on the n-sheeted Riemann surface is characterized by the winding numbers r_k, with

$$\int_{k\beta}^{(k+1)\beta} d\tau\, \partial_\tau \phi(\tau, x) = 2\pi r_k, \quad 0 < \mathrm{x} < \ell \tag{A3}$$

and

$$\int_0^{n\beta} d\tau\, \partial_\tau \phi(\tau, x) = 2\pi \sum_k r_k, \quad \ell < \mathrm{x} < L \tag{A4}$$

Here we are assuming that the field $\phi(\tau, x)$ has no vortices in the (τ, x) plane, also known as phase-slips. Such phase-slips have an action proportional to L^{d-1} and are, hence, exponentially suppressed.[6] Ordinary spatial vortices are also exponentially suppressed as $T \to 0$.

Any field in the sector with a given set of winding numbers may be written as

$$\phi(x) = \phi_r(x) + \delta\phi(x) \tag{A5}$$

Here, ϕ_r is a reference field carrying winding numbers $\{r_k\}$ and $\delta\phi(x)$ is a field with all winding numbers equal to zero. We choose ϕ_r to satisfy

$$\partial^2 \phi_r = 0 \tag{A6}$$

We will discuss the solution to this equation shortly. Then

$$S[\phi] = S[\phi_r] + S[\delta\phi] \tag{A7}$$

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[6] The phase-slips are also often suppressed by lattice symmetries.

Thus, we may perform the integral over $\delta\phi$, to obtain

$$Z_n = Z_n^{r=0} W_n \tag{A8}$$

with

$$W_n = \sum_r e^{-S[\phi_r]} \tag{A9}$$

We have already computed the partition function $Z_n^{r=0}$ in the zero winding number sector - it is given by Eq. (3.41). It remains to compute $S[\phi_r]$.

Let us solve Eq. (A6) in a sector with winding numbers $\{r_k\}$. We take $\phi(x)$ to be independent of the variables along the subsystem boundary and focus on the (x, τ) plain. We again introduce fields ϕ_k via Eq. (3.9). As before the fields are glued along the branch-cuts at $\tau = k\beta$ using Eq. (3.10). However, to introduce the winding numbers instead of Eq. (3.11) we now impose,

$$\phi_k(\tau) = \phi_{k+1}(\tau - \beta) + 2\pi r_k \tag{A10}$$

As before, we work in the limit $\beta \gg L$, such that the τ variable in each sheet runs from $(-\infty, \infty)$. The solution to the Laplace equation with the above boundary conditions is then given by Eq. (3.17). In the present case, we may set $C = 0$ (a finite C can be absorbed into $\delta\phi$). Using the expansion (3.19) and imposing boundary conditions (3.10), (A10) we find,

$$c_k^- = -c_{k-1}^+ \tag{A11}$$

and

$$c_k^+ = -\frac{L}{\beta} r_k + O\left(\frac{L^2}{\beta^2}\right) \tag{A12}$$

Note that due to the relation (A11), the constraint (3.18) is automatically satisfied.

We are now ready to calculate the action $S[\phi_r]$ in each winding number sector. We may write,

$$S[\phi_r] = \frac{\rho_s L^{d-1}}{2} \int_0^{n\beta} d\tau \int_0^L dx (\partial_\mu \phi)^2 = \frac{\rho_s L^{d-1}}{2} \sum_k \int_{-\beta/2}^{\beta/2} d\tau \int_0^L dx (\partial_\mu \phi_k)^2 \tag{A13}$$

Here we've broken up the integral into contributions from each sheet. We now integrate by parts and use Eq. (A6) to reduce Eq. (A13) to an integral over the boundary of each sheet,

$$\begin{aligned} S[\phi_r] &= \frac{\rho_s L^{d-1}}{2} \sum_k \int_k dS_\mu \, \phi_k \partial_\mu \phi_k \\ &= \frac{\rho_s L^{d-1}}{2} \sum_k \Bigg(\int_0^L dx((\phi_k \partial_\tau \phi_k)(x, \beta/2) - (\phi_k \partial_\tau \phi_k)(x, -\beta/2)) \end{aligned} \tag{A14}$$

$$- \int_0^\ell ((\phi_k \partial_\tau \phi_k)(x, 0^+) - (\phi_k \partial_\tau \phi_k)(x, 0^-)) \Bigg) \tag{A15}$$

The contribution (A15) comes from the cuts at $\tau = k\beta$ and vanishes by Eq. (3.10). On the other hand, using Eq. (A10), the contribution (A14) becomes

$$S[\phi_r] = \frac{\rho_s L^{d-1}}{2} \sum_k (2\pi r_k) \int_0^L dx\, \partial_\tau \phi_k(\mathrm{x}, \beta/2) = \frac{\rho_s L^{d-1}}{2} \sum_k (2\pi r_k) \left(-2\pi c_k^+ \right) \approx \frac{\rho_s L^d}{2\beta} \sum_k (2\pi r_k)^2 \tag{A16}$$

where we've used Eqs. (3.19),(A12) and dropped a correction of order L/β in the last step. Therefore, the winding number factor W_n in Eq. (A9) satisfies,

$$W_n \approx \nu^n (e^{-2\pi^2 \rho_s L^d T}) \tag{A17}$$

with $\nu(q) = \sum_{r=-\infty}^{\infty} q^{r^2}$ - the Jacobi theta function. Observe that

$$\frac{W_n}{W_1^n} = 1 + O(L/\beta) \tag{A18}$$

Hence, the winding number contribution does not to leading order modify our previous result for the entanglement entropy Eq. (3.43) for $T \ll 1/L$. On the other hand, the partition function on the n-sheeted Riemann surface is modified by the inclusion of W_n,

$$\log Z_n \overset{N=2}{=} \frac{1}{2} \left(n\beta\Lambda + \log(n\rho_s L^d/\beta) - (n-1)\log(2\beta\Lambda) + 2n\log\nu(\exp(-2\pi^2\rho_s L^d T)) + \log(2\pi) \right) \tag{A19}$$

In the limit, $T \gg \Delta_{\text{tower}} \sim \frac{1}{\rho_s L^d}$, $W_n \to 1$ and the partition function (A19) reduces to our previous result (3.41) as expected. However, using the identity $\nu(\exp(-\alpha)) = \sqrt{\pi/\alpha}\,\nu(\exp(-\pi^2/\alpha))$, we may also rewrite Eq. (A19) as,

$$\log Z_n = \frac{1}{2} \left(n\beta\Lambda - (n-1)\log(4\pi\rho_s L^d\Lambda) + \log n + 2n\log\nu(\exp(-1/(2\rho_s L^d T))) \right) \tag{A20}$$

so that for $T \ll \Delta_{\text{tower}}$,

$$\log Z_n \to \frac{1}{2} \left(n\beta\Lambda - (n-1)\log(4\pi\rho_s L^d\Lambda) + \log n \right) \tag{A21}$$

Thus, we see that the singular logarithmic dependence of (3.41) in the limit $T \to 0$ disappears once the sum over winding numbers is performed.

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